Modeling The Business Cycle
Part II - Net Income And Investment

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In this white paper we will build a model that calculates net income and investment for a company whose revenues are correlated with the business cycle. To that end we will work through the following hypothetical problem from Part I...

Our Hypothetical Problem

In Part I we were tasked with forecasting revenue for ABC Company. The table below presents ABC Company's go-forward model assumptions...

Table 1: Model Assumptions

Description	Balance	Notes
Annualized revenue at time zero (in thousands)	$10,000	Current revenue annualized
Annualized revenue growth rate (%)	5.00	Discrete-time secular growth rate (RGR)
Annualized revenue volatility (%)	25.00	Secular growth rate standard deviation
Assets as a percent of annualized revenue (%)	60.00	Total assets divided by annualized revenue
Return on assets (%)	13.50	After-tax ROA
Peak-to-trough change in revenue (%)	50.00	Excludes secular growth rate
Business cycle length in months	60	Peak-to-peak or trough-to-trough

We are tasked with answering the following questions:

Question 1: What is cumulative net income in year 2?

Question 2: What is cumulative net investment in year 2?

Question 3: Graph net income, investment and the earnings growth rate over the first 10 years.

Annualized Revenue

In Part I we defined the variable R_t to be cyclical random annualized revenue at time t, the variable g to be the secular revenue growth rate, the variable Δ to be the business cycle amplitude, the variable ω to be the length of the business cycle in years, and the variable ϕ to be the current position in the business cycle in years. The equation for expected annualized revenue is... [1]

$$\mathbb{E}\Bigg[R_t\Bigg] = R_0\Bigg(1+\Delta\,\sin(\beta\,\phi)\Bigg)^{-1}\text{Exp}\Bigg\{\lambda\,t\Bigg\}\Bigg(1+\Delta\,\sin(\beta\,(t+\phi))\Bigg)\;\text{...where...}\;\lambda=\ln(1+g)\;\text{...and...}\;\beta=\frac{2\,\pi}{\omega} \quad (1)$$

Note that we can rewrite Equation (1) above as...

$$\mathbb{E}\Bigg[R_t\Bigg] = R_0\Bigg(1+\Delta\,\sin(\beta\,\phi)\Bigg)^{-1}\Bigg(\text{Exp}\Bigg\{\lambda\,t\Bigg\}+\Delta\,\text{Exp}\Bigg\{\lambda\,t\Bigg\}\,\sin(\beta\,(t+\phi))\Bigg) \quad (2)$$

Using Appendix Equation (20) below we can rewrite Equation (2) above as...

$$\mathbb{E}\Bigg[R_t\Bigg] = R_0\Bigg(1+\Delta\,\sin(\beta\,\phi)\Bigg)^{-1}\Bigg(E_1+\Delta\,E_2\Bigg) \quad (3)$$

Using Equation (3) above and Appendix Equations (24) and (25) below the equation for the derivative of expected annualized revenue with respect to time is...

$$\frac{\delta}{\delta t}\,\mathbb{E}\Big[R_t\Big] = R_0\Big(1+\Delta\,\sin(\beta\,\phi)\Big)^{-1}\Big(\frac{\delta}{\delta t}\,E_1+\Delta\,\frac{\delta}{\delta t}\,E_2\Big)$$
$$= R_0\Big(1+\Delta\,\sin(\beta\,\phi)\Big)^{-1}\,\mathrm{Exp}\Big\{\lambda\,t\Big\}\Big(\lambda+\Delta\,\lambda\,\sin(\beta\,(t+\phi))+\Delta\,\beta\,\cos(\beta\,(t+\phi))\Big) \tag{4}$$

Assets

We will define the variable A_t to be total assets at time t and the variable ϵ to be the ratio of total assets to annualized revenue. Using Equation (3) above the equation for expected total assets at time t from the perspective of time zero is...

$$\mathbb{E}\Big[A_t\Big] = \epsilon\,\mathbb{E}\Big[R_t\Big] = \epsilon\,R_0\Big(1+\Delta\,\sin(\beta\,\phi)\Big)^{-1}\Big(E_1+\Delta\,E_2\Big) \tag{5}$$

Using Equation (4) above the equation for the derivative of total assets with respect to time is...

$$\frac{\delta}{\delta t}\,\mathbb{E}\Big[A_t\Big] = \epsilon\,\frac{\delta}{\delta t}\,\mathbb{E}\Big[R_t\Big]\delta t = \epsilon\,R_0\Big(1+\Delta\,\sin(\beta\,\phi)\Big)^{-1}\,\mathrm{Exp}\Big\{\lambda\,t\Big\}\Big(\lambda+\Delta\,\lambda\,\sin(\beta\,(t+\phi))+\Delta\,\beta\,\cos(\beta\,(t+\phi))\Big) \tag{6}$$

We will define the variable $M_{a,b}$ to be cumulative investment over the time interval $[a, b]$. Using Equation (6) above the equation for cumulative investment is...

$$M_{a,b} = \int_a^b \frac{\delta}{\delta t}\,\mathbb{E}\Big[A_t\Big]\,\delta t$$
$$= \int_a^b \epsilon\,R_0\Big(1+\Delta\,\sin(\beta\,\phi)\Big)^{-1}\Big(\lambda\,\mathrm{Exp}\Big\{\lambda\,t\Big\}+\Delta\,\beta\,\mathrm{Exp}\Big\{\lambda\,t\Big\}\cos(\beta\,(t+\phi))+\Delta\,\lambda\,\mathrm{Exp}\Big\{\lambda\,t\Big\}\sin(\beta\,(t+\phi))\Big)\delta t$$
$$= \epsilon\,R_0\Big(1+\Delta\,\sin(\beta\,\phi)\Big)^{-1}\Big(\lambda\int_a^b \mathrm{Exp}\Big\{\lambda\,t\Big\}\,\delta t+\Delta\,\beta\lambda\int_a^b \mathrm{Exp}\Big\{\lambda\,t\Big\}\cos(\beta\,(t+\phi))\,\delta t+\Delta\,\lambda\int_a^b \mathrm{Exp}\Big\{\lambda\,t\Big\}\sin(\beta\,(t+\phi))\,\delta t\Big) \tag{7}$$

Using Appendix Equations (21), (22) and (23) below we can rewrite Equation (7) above as...

$$M_{a,b} = \epsilon\,R_0\Big(\lambda\,I(a,b)_1+\Delta\,\beta\,I(a,b)_3+\Delta\,\lambda\,I(a,b)_2\Big) \tag{8}$$

Net Income

We will define the variable N_t to be annualized net income at time t and the variable π to be the after-tax return on assets. Using Equation (5) above the equation for expected annualized net income at time t is...

$$N_t = \pi\,\mathbb{E}\Big[A_t\Big] = \epsilon\,\pi\,\mathbb{E}\Big[R_t\Big] \tag{9}$$

Using Equation(6) above the derivative of annualized net income with respect to time is...

$$\frac{\delta}{\delta t}\,N_t = \pi\,\frac{\delta}{\delta t}\,\mathbb{E}\Big[A_t\Big] = \epsilon\,\pi\,\frac{\delta}{\delta t}\,\mathbb{E}\Big[R_t\Big] \tag{10}$$

We will define the variable EGR_t to be the annualized earnings growth rate at time t. Using Equations (1), (4), (9) and (10) above the equation for the earnings growth rate is...

$$EGR_t = \frac{\frac{\delta}{\delta t}N_t}{N_t} = \frac{\lambda+\Delta\,(\beta\,\cos(\beta\,(t+\phi))+\lambda\,\sin(\beta\,(t+\phi)))}{1+\Delta\,\sin(\beta\,(t+\phi))} \tag{11}$$

We will define the variable $N_{a,b}$ to be cumulative net income realized over the time interval $[a,b]$. Using Equations (5) and (9) above the equation for cumulative net income is...

$$N_{a,b} = \int_a^b N_t \, \delta t = \int_a^b \pi \, \mathbb{E}\bigg[A_t\bigg] \, \delta t = \int_a^b \pi \, \epsilon \, R_0 \bigg(1 + \Delta \, \sin(\beta \, (t + \phi))\bigg) \text{Exp}\bigg\{\lambda \, t\bigg\} \, \delta t \tag{12}$$

Note that we can rewrite Equation (12) above as...

$$N_{a,b} = \pi \, \epsilon \, R_0 \bigg(\int_a^b \text{Exp}\bigg\{\lambda \, t\bigg\} \, \delta t + \Delta \int_a^b \text{Exp}\bigg\{\lambda \, t\bigg\} \, \sin(\beta \, (t + \phi)) \, \delta t \bigg) \tag{13}$$

Using Appendix Equations (21) and (22) below we can rewrite Equation (13) above as...

$$N_{a,b} = \pi \, \epsilon \, R_0 \bigg(I(a,b)_1 + \Delta \, I(a,b)_2 \bigg) \tag{14}$$

The Answers To Our Hypothetical Problem

Using the data in Table 1 above we will make the following earnings-related variable definitions...

$$\alpha = \lambda = \ln(1 + 0.05) = 0.0488 \ \text{...and...} \ \pi = 0.1350 \ \text{...and...} \ \epsilon = 0.60 \tag{15}$$

Using the data in Table 1 above we will make the following business cycle-related variable definitions...

$$\omega = \frac{60}{12} = 5.00 \ \text{...and...} \ \phi = \frac{15}{12} = 1.25 \ \text{...and...} \ \beta = \frac{2\,\pi}{5.00} = 1.2566 \ \text{...and...} \ \Delta = \frac{0.50}{2} = 0.25 \tag{16}$$

Using Equations (15) and (16) above and the appendix equations below the values of the following integrals are...

$$I(1,2)_1 = 1.07604 \ \text{...and...} \ I(1,2)_2 = -0.31609 \ \text{...and...} \ I(1,2)_3 = -0.95572 \tag{17}$$

Question 1: What is cumulative net income in year 2?

Using Equations (14), (15), (16), and (17) above the answer to the question is...

$$N_{2,3} = 0.1350 \times 0.60 \times 10,000,000 \times \bigg(1 + 0.25 \times \sin(1.2566 \times 1.25)\bigg)^{-1} \times \bigg(1.07604 + 0.25 \times -0.31609\bigg) = 646,000 \tag{18}$$

Question 2: What is cumulative net investment in year 2?

Using Equations (8), (15), (16), and (17) above the answer to the question is...

$$\begin{aligned} M_{2,3} = 0.60 \times 10,000,000 \times \bigg(1 + 0.25 \times \sin(1.2566 \times 1.25)\bigg)^{-1} \times \bigg(&0.0488 \times 1.07604 + 0.25 \times 1.2566 \\ \times -0.95572 + 0.25 \times 0.0488 \times -0.31609\bigg) = -1,208,000 \end{aligned} \tag{19}$$

Note that during year 2 annualized revenue was decreasing (due to cyclicallity) so assets were also decreasing such that net investment was negative.

Question 3: Graph net income, investment and the earnings growth rate over the first 10 years.

Figure 1: Net Income and Investment

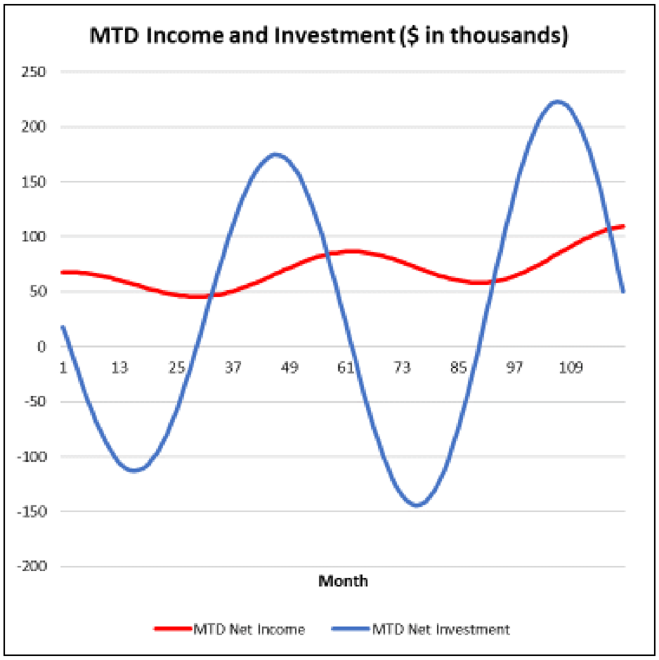
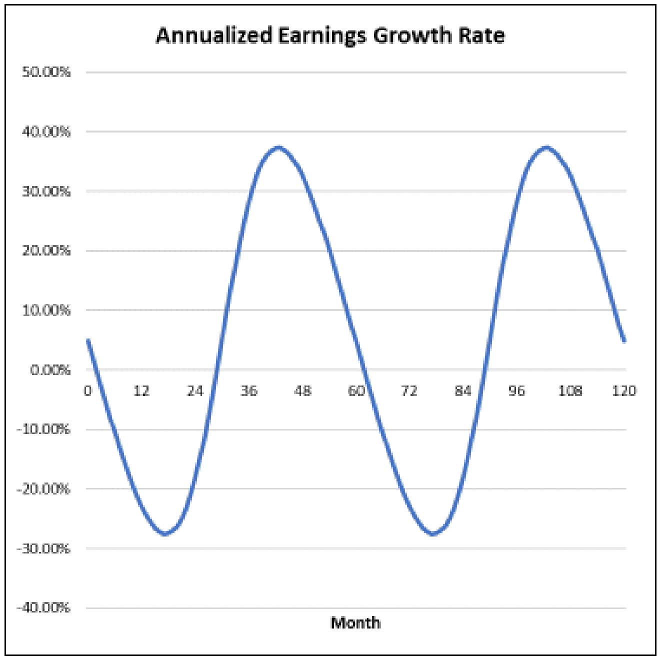


Figure 2: Earnings Growth Rate



Appendix

A. We will define the following equations... [2]

$$E_1 = \text{Exp}\left\{\alpha\, t\right\} \text{ ...and... } E_2 = \text{Exp}\left\{\alpha\, t\right\}\sin(\beta\,(t+\phi)) \text{ ...and... } E_3 = \text{Exp}\left\{\alpha\, t\right\}\cos(\beta\,(t+\phi)) \tag{20}$$

B. Using the first equation in Equation (20) above we will make the following integral definition... [2]

$$I(a,b)_1 = \int_a^b E_1\,\delta t = \text{Exp}\left\{\alpha\, t\right\}\alpha^{-1}\Big[_a^b \tag{21}$$

C. Using the second equation in Equation (20) above we will make the following integral definition... [2]

$$I(a,b)_2 = \int_a^b E_2\,\delta t = \text{Exp}\left\{\alpha\, t\right\}\Big(\alpha\,\sin(\beta\,(t+\phi)) - \beta\,\cos(\beta\,(t+\phi))\Big)\Big(\alpha^2+\beta^2\Big)^{-1}\Big[_a^b \tag{22}$$

D. Using the third equation in Equation (20) above we will make the following integral definition... [2]

$$I(a,b)_3 = \int_a^b E_3\,\delta t = \text{Exp}\left\{\alpha\, t\right\}\Big(\beta\,\sin(\beta\,(t+\phi)) + \alpha\,\cos(\beta\,(t+\phi))\Big)\Big(\alpha^2+\beta^2\Big)^{-1}\Big[_a^b \tag{23}$$

E. Using the first equation in Equation (20) above we will make the following derivative definition... [2]

$$\frac{\delta}{\delta t}E_1 = \alpha\,\text{Exp}\left\{\alpha\, t\right\} \tag{24}$$

F. Using the second equation in Equation (20) above we will make the following derivative definition... [2]

$$\frac{\delta}{\delta t}E_2 = \text{Exp}\left\{\alpha\, t\right\}\Big(\alpha\,\sin(\beta\,(t+\phi)) + \beta\,\cos(\beta\,(t+\phi))\Big) \tag{25}$$

G. Using the third equation in Equation (20) above we will make the following derivative definition... [2]

$$\frac{\delta}{\delta t}E_3 = \text{Exp}\left\{\alpha\, t\right\}\Big(\alpha\,\cos(\beta\,(t+\phi)) - \beta\,\sin(\beta\,(t+\phi))\Big) \tag{26}$$

References

[1] Gary Schurman, *Modeling The Business Cycle - Part I,* October, 2020.

[2] Gary Schurman, *Modeling The Business Cycle - Mathematical Supplement,* October, 2020.